MATH 732: CUBIC HYPERSURFACES

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1. Basic cohomological invariants of cubics

These notes are based on [Huy23, §1.1] and [Huy23, §1.3]. Please see the disclaimer section.

Let k be an algebraically closed field. Here we work out the cohomological invariants of smooth cubics over k. Sometimes we will assume $k = \mathbf{C}$, and we will try to point that out.

Theorem 1.1 (Lefschetz hyperplane theorem)**.** *Suppose that $X \subseteq Y$ is a smooth ample hypersurface in a smooth, $(n+1)$-dimensional projective k-variety Y. If $k = \mathbf{C}$:*

(1) $\mathrm{H}^m(Y,\mathbf{Z}) \to \mathrm{H}^m(X,\mathbf{Z})$ is an isomorphism for $m < n$ and injective for $m = n$.
(2) $\mathrm{H}_m(X,\mathbf{Z}) \to \mathrm{H}_m(X,\mathbf{Z})$ is an isomorphism for $m < n$ and surjective for $m = n$.
(3) $\pi_m(X) \to \pi_m(Y)$ is an isomorphism for $m < n$ and surjective for $m = n$.

(The analogous results hold when k is arbitrary for ℓ-adic cohomology.)

So, by Poincaré duality, the cohomology of Y determines all but the n-th (middle) cohomology group of X! For $Y = \mathbf{P}^{n+1}$ (and say $k = \mathbf{C}$) we have:

$$\mathrm{H}^m(\mathbf{P}^{n+1},\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } m \text{ even and } 0 \le m \le 2n+2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus for a smooth (complex) cubic hypersurface $X \subseteq \mathbf{P}^{n+1}$ we have:

$$\mathrm{H}^m(X,\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } m \text{ even and } (0 \le m < n \text{ or } n < m \le 2n), \\ 0 & \text{otherwise.} \end{cases}$$

What remains to resolve is the middle cohomology group $\mathrm{H}^n(X,\mathbf{Z})$.

Remark 1.2 (Torsion free cohomology)**.** Letting X be a smooth complex cubic (or degree d) hypersurface, by the universal coefficients theorem there is an exact sequence:

$$0 \to \mathrm{Ext}^1(\mathrm{H}_{m-1}(X,\mathbf{Z}),\mathbf{Z}) \to \mathrm{H}^m(X,\mathbf{Z}) \to \mathrm{Hom}(\mathrm{H}_m(X,\mathbf{Z}),\mathbf{Z}) \to 0.$$

Thus the torsion in $\mathrm{H}^n(X,\mathbf{Z})$ is determined by the torsion in $\mathrm{H}_n(X,\mathbf{Z})$ which is trivial here. In particular, this implies $\mathrm{H}^n(X,\mathbf{Z})$ is determined as a group by its rank. This can be determined by the *topological Euler characteristic* of X.

The Euler characteristic $e(X)$ of a smooth complex hypersurface $X \subseteq \mathbf{P}^{n+1}$ of degree d is

$$e(X) = \begin{cases} n + b_n(X) & \text{if } n \text{ is even} \\ n+1-b_n(X) & \text{if } n \text{ is odd.} \end{cases}$$

(here $b_n(X) = \mathrm{rk}(\mathrm{H}^n(X,\mathbf{Z}))$). It is nice to phrase this in terms of the *primitive Betti number.*

Definition 1.3. The *primitive n-th cohomology of a smooth n-dimensional complex projective variety $X \subseteq \mathbf{P}^N$* is

$$\mathrm{H}^n(X,\mathbf{Z})_{\mathrm{prim}} = \ker(h \cup -\colon \mathrm{H}^n(X,\mathbf{Z}) \to \mathrm{H}^{n+2}(X,\mathbf{Z}).$$

Here h is the restriction of the hyperplane class of $\mathbf{P}^N$ to $\mathrm{H}^2(X,\mathbf{Z})$. This can be defined similarly for other degrees of cohomology (see Voisin). We set $b_n(X)_{\mathrm{prim}} = \mathrm{rk}(\mathrm{H}^n(X,\mathbf{Z})_{\mathrm{prim}})$.

In terms of primitive cohomology we have the relation:

$$b_n(X)_{\mathrm{prim}} = (-1)^n(e(X) - (n+1)).$$

By the Poincaré—Hopf theorem, the Euler characteristic of a degree d hypersurface is computed as:

$$e(X) = \int_X c_n(X), \qquad \text{(Poincaré—Hopf)}$$

i.e. it is the degree of the top Chern class of X.

Remark 1.4. If you have not seen Chern classes before, now is a good time to learn them! I can give you references or an outline of the ideas during office hours. There are many good exercises about Chern classes too, that help one get a feel for their usefulness.

Lemma 1.5. *If $X \subseteq \mathbf{P}^{n+1}$ is a degree d hypersurface, then*

(1) $c_n(X) = \frac{1}{d^2}\left((1-d)^{n+2} + d(n+2) - 1\right) h^n$,
(2) $e(X) = \frac{1}{d}\left((1-d)^{n+2} + d(n+2) - 1\right)$, *and thus*
(3) $b_n(X)_{\mathrm{prim}} = \frac{(-1)^n}{d}\left(d - 1 + (1-d)^{n+2}\right)$.

Proof. The total Chern class $c(X)$ can determined by the Euler sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(h)^{\oplus(n+2)} \to T_{\mathbf{P}}|_X \to 0 \qquad \text{(Euler sequence)}$$

and the normal bundle sequence:

$$0 \to T_X \to T_{\mathbf{P}}|_X \to N_X = \mathcal{O}_X(d \cdot h) \to 0 \qquad \text{(Normal bundle sequence)}$$

using the splitting principle. This gives:

$$\begin{aligned} c(X) = c(T_X) &= c(T_{\mathbf{P}}|_X)c(\mathcal{O}_X(d \cdot h))^{-1} \\ &= c(\mathcal{O}_X(h)^{\oplus(n+2)})/(1 + dh) \\ &= (1 + h)^{n+2}(1 - dh + (dh)^2 - \dots) \end{aligned}$$

The class $c_n(X)$ is computed by isolating the term with h^n. We have $\int_X h^n = d$ by the projection formula. □

Example 1.6. So, for example, the topological Euler characteristic of a degree d plane curve is $((1 - d)^3 + 3d - 1)/d = 3d - d^2$.

Remark 1.7 (Middle Betti number of cubics)**.** The primitive Betti number of a cubic hypersurface is thus:

$$b_n(X)_{\text{prim}} = \frac{(-1)^n}{3}(2 + (-2)^{n+2}).$$

For small n we have

n	1	2	3	4	5
$b_n(X)_{\text{prim}}$	2	6	10	22	42

Remark 1.8. It remains to determine the structure of $\mathrm{H}^*(X, \mathbf{Z})$ as a ring. This can be found in the reference [Huy23].

Remark 1.9. We can use comparison theorems to prove similar results for étale cohomology of hypersurfaces over the complex numbers. For hypersurfaces over algebraically closed fields in positive characteristic, we can again use comparison theorems to compute their cohomology using that they spread out to smooth hypersurfaces in characteristic zero.

Exercise 1. *Assume $X \subseteq \mathbf{P}^{n+1}$ is a smooth hypersurface of degree $d > 1$ and $\mathbf{P}^\ell \subseteq X$ is a linear subspace contained in X. Show that $\ell \leq n/2$.*

Exercise 2. *Conversely, prove that if $\ell \leq n/2$ then there exist smooth hypersurfaces of every degree that contain $\mathbf{P}^\ell$. For which d does every degree d hypersurface in $\mathbf{P}^{n+1}$ contain a $\mathbf{P}^\ell$?*

Exercise 3. *Assume that* $X \subseteq \mathbf{P}^{n+1}$ *is a smooth hypersurface of degree* $d > 1$. *Let* $h \in \mathrm{H}^2(X, \mathbf{Z})$ *represent the restriction of the hyperplane class. Prove that*

$$\mathrm{H}^{2k}(X, \mathbf{Z}) = \begin{cases} \mathbf{Z}h^k & 0 < 2k < n \\ \mathbf{Z}h^k/d & n < 2k < d \end{cases}$$

References

[Huy23] Daniel Huybrechts. *The geometry of cubic hypersurfaces*, volume 206 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2023.